Universidad Simón Bolívar
Departamento de Matemáticas
Puras y Aplicadas

Matemáticas VII (MA3111)
1^{er} Examen Parcial
Enero-Marzo 2020

Sección 3-4

Primer Parcial

1. a) Hallar el valor de la constante A para que $y(x) = A\delta'(x)$ sea solución de $xu'(x) + u'(x) = \delta''(x)$, $0 < x < \infty$.

b) Halle los valores de a,b para que $f(x) = a\delta(x) - b\delta'(x)$ sea solución de $xf(x) = \delta(x)$.

2. Resolver:

$$xy'(x) + 2\int_0^x y(t)\operatorname{sen}(x-t)\,dt = 0 \quad \text{con} \quad y(0) = 1;\ x > 0$$

3. Sea $g(x) = \mathrm{H}(x)\cos x$. Halle $f(x)$ generalizada que satisfaga $(f * g)(x) = \mathrm{H}(x-1)[x - 1 - \operatorname{sen}(x-1)] + \mathrm{H}(x)\displaystyle\int_0^{\infty} xe^{-2x}\operatorname{sen} x\,dx$.

Soluciones:

Pregunta 1.

Parte a). Tenemos $y(x) = \mathrm{A}\delta'(x)$

Luego, $y'(x) = \delta''(x)$, $y''(x) = \delta'''(x)$ y sea $\oslash(x)$ una función de prueba.

Luego, $xu''(x) + u'(x) = A\delta''(x)$

$< x\,A\,\delta'''(x) \,/\, \oslash(x) > + < A\,\delta''(x) \,/\, \oslash(x) > = < \delta''(x) \,/\, \oslash(x) >$

$A < x\,\delta'''(x) \,/\, \oslash(x)x > + A < \delta''(x) \,/\, \oslash(x) > = < \delta''(x) \,/\, \oslash(x) >$

$(-1)^3\,A < x\,\delta(x) \,/\, (\oslash(x)x)''' > + (-1)^2\,A < \delta(x) \,/\, \oslash''(x) > = \ (-1)^2 < \delta''(x) \,/\, \oslash''(x) >$

$-A < x\,\delta(x) \,/\, (\oslash(x)x)''' > + A < \delta(x) \,/\, \oslash''(x) > = < \delta''(x) \,/\, \oslash''(x) > \qquad \text{(I)}$

Calculo de $(\oslash(x)x)'''$:

- $(\oslash(x)x)' = \oslash'(x)x + \oslash(x)$
- $(\oslash(x)x)'' = \oslash''(x)x + 2\,\oslash'(x)$
- $(\oslash(x)x)''' = \oslash'''(x)x + 2\,\oslash''(x) + 2\,\oslash''(x) = \oslash'''(x)x + 3\,\oslash''(x)$

Sustituyendo en (I) la ecuación generalizada:

$$-A < x\,\delta(x)\,/\,[\varnothing'''(x)x + 3\,\varnothing''(x)] > +A < \delta(x)\,/\,\varnothing''(x) > = < \delta''(x)\,/\,\varnothing''(x) >$$

Así,

$$-A\,[\cancelto{0}{\varnothing'''(0)(0)} + 3\,\varnothing''(0)] + A\,(\varnothing''(0)) = \varnothing''(0)$$

$$-3A\,(\varnothing''(0)) + A\,(\varnothing''(0)) = \varnothing''(0)$$

$$-2A\,(\varnothing''(0)) = \varnothing''(0) \;\rightarrow\; -2A < \delta''(x)\,/\,\varnothing(x) > = < \delta''(x)\,/\,\varnothing(x) >$$

$$\rightarrow\; -2A\delta''(x) = \delta''(x)$$

Luego por Coef. Indeterminados:

$$-2A = 1 \qquad \rightarrow \boxed{A = -\frac{1}{2}}$$

Parte b). Tenemos $f(x) = a\delta(x) - b\delta'(x)$, función generalizada, y la ecuación $xf(x) = \delta(x)$ (I), sustituimos *f(x)* en (I):

$$x\,(a\delta(x) - b\delta'(x)) = \delta(x)$$

$$ax\delta(x) - bx\delta'(x) = \delta(x)$$

Aplicamos la Transf. Laplace en ambos lados de la ecuación:

$$\mathscr{L}\,(ax\delta(x))\,(z) - \mathscr{L}\,(bx\delta'(x))\,(z) = \mathscr{L}\,(\delta(x))\,(z)$$

$$-a\,\cancelto{0}{\frac{d(1)}{dz}} + b\,\frac{d(z)}{dz} = 1 \;\rightarrow\; \boxed{b = 1}\ \text{y } a \text{ puede ser una constante arbitraria.}$$

Así,

$$\boxed{f(x) = a\delta(x) - \delta'(x),\;\; a = cte}$$

Pregunta 2. Tenemos $xy'(x) + 2\displaystyle\int_0^x y(t)\,\mathrm{sen}(x-t)\,dt = 0, \quad y(0) = 1 \;\therefore\; x > 0$

Multiplicamos ambos lados de la ecuación por $\mathrm{H}(x)$:

$$xy'(x)\,\mathrm{H}(x) + 2\,\underbrace{\mathrm{H}(x)\int_0^x y(t)\,\mathrm{sen}(x-t)\,dt}_{[\mathrm{I}]} = 0$$

Para [I]:

Sabemos que si $\mathrm{F}(x) = f(x)\,\mathrm{H}(x-a)$ y $\mathrm{G}(x) = g(x)\,\mathrm{H}(x-b)$, la convolución entre $\mathrm{F}(x)$ y $\mathrm{G}(x)$ vendrá dada por:

$$\mathrm{H}(x-a-b)\int_{b}^{x-a} f(t)\cdot g(x-t)\,dt = (\mathrm{F}*\mathrm{G})(x)$$

En nuestro caso $\mathrm{F}(x) = \mathrm{H}(x)\,y(x)$ y $\mathrm{G}(x) = \mathrm{H}(x)\mathrm{gen}(x), \quad a = b = 0$

Así, podemos reescribir la ecuación inicial como:

$$xy'(x)\,\mathrm{H} + 2\left(\mathrm{H}(x)y(x) * \mathrm{H}(x)\,\mathrm{sen}(x)\right) = 0 \qquad \text{(II)}$$

Luego, sea $u(x) = y(x)\,\mathrm{H}(x)$

$$u'\mathrm{gen}(x) = y'(x)\,\mathrm{H}(x) + \delta(x) \;\rightarrow\; y'(x)\,\mathrm{H}(x) = u'\mathrm{gen}(x) - \delta(x)$$

Sustituyendo en (II):

$$x\left(u'\mathrm{gen}(x) - \delta(x)\right) + 2\left(u(x) * \mathrm{H}(x)\,\mathrm{sen}(x)\right) = 0$$

$$xu'\mathrm{gen}(x) - x\delta(x) + 2\left(u(x) * \mathrm{H}(x)\,\mathrm{sen}(x)\right) = 0$$

Aplicamos la Transf. de Laplace en ambos lados de la ecuación, tomando $u(x) \xrightarrow{\mathscr{L}} \mathrm{F}(z)$

$$\mathscr{L}\left(xu'\mathrm{gen}(x)\right)(z) - \mathscr{L}\left(x\delta(x)\right)(z) + 2\,\mathscr{L}\left(u(x) * \mathrm{H}(x)\,\mathrm{sen}(x)\right)(z) = 0$$

$$\mathscr{L}\left(xu'\mathrm{gen}(x)\right)(z) - \mathscr{L}\left(x\delta(x)\right)(z) + 2\,\mathscr{L}\left(u(x)\right)(z) * \mathscr{L}\left(\mathrm{H}(x)\,\mathrm{sen}(x)\right)(z) = 0$$

$$-\frac{d(z\mathrm{F}(z))}{dz} + \frac{d(z)}{dz} + 2\mathrm{F}(z)\frac{1}{z^2+1} = 0$$

$$-z\mathrm{F}(z) - \mathrm{F}(z) + \frac{2\mathrm{F}(z)}{z^2+1} = 0 \;\rightarrow\; \mathrm{F}(z)\left(\frac{2}{z^2+1} - 1\right) = z\mathrm{F}(z)$$

$$\mathrm{F}(z)\left(\frac{2-z^2-1}{z^2+1}\right) = z\mathrm{F}(z) \;\rightarrow\; \mathrm{F}(z)\left(\frac{1-z^2}{z^2+1}\right) = z\mathrm{F}(z),\; \mathrm{F}(z) = \frac{d\mathrm{F}(z)}{dz}$$

Así:

$$\frac{d\mathrm{F}(z)}{\mathrm{F}(z)} = \frac{(1-z^2)\,dz}{z(z^2+1)} \;\rightarrow\; \frac{1-z^2}{z(z^2+1)} = \frac{\mathrm{A}}{z} + \frac{\mathrm{B}z+\mathrm{C}}{1+z^2}$$

$$1 - z^2 = \mathrm{A}(1-z^2) + (\mathrm{B}z+\mathrm{C})z$$

- $z = 0 \;\rightarrow\; \boxed{\mathrm{A} = 1}$

- $z = 1 \ \rightarrow \ 2 + \mathrm{B} + \mathrm{C} = 0 \qquad (1)$
- $z = -1 \ \rightarrow \ 2 + \mathrm{B} - \mathrm{C} = 0 \qquad (2)$

Sumando (1) y (2) $\rightarrow \ 4 + 2\mathrm{B} = 0 \ \rightarrow \ \boxed{\mathrm{B} = -2}\,, \quad \boxed{\mathrm{C} = 0}$

Luego,

$$\frac{d\mathrm{F}(z)}{\mathrm{F}(z)} = \left(\frac{1}{z} - \frac{2z}{1 + z = 2}\right) dz \ \rightarrow \ \mathrm{Ln}\,(\mathrm{F}(z)) = \mathrm{Ln}(z) + \mathrm{Ln}(1 + z^2) + \mathrm{Ln}(k)$$

$\mathrm{F}(z) = \dfrac{kz}{1 + z^2}$, aplicando la Transf. Inversa , donde $\mathrm{F}(z) \xrightarrow{\mathscr{L}^{-1}} u(x)$

Así,

$$\mathscr{L}^{-1}\left(\mathrm{F}(z)\right)(x) = \mathscr{L}^{-1}\left(\frac{kz}{1 + z^2}\right)(x) \ \rightarrow \ u(x) = k\mathrm{H}(x)\cos(x) = y(x)\mathrm{H}(x)$$

$$y(x) = k\cos(x) \ \rightarrow \ y(0) = 1 = k$$

Finalmente,

$$\boxed{y(x) = \cos(x)}$$

Pregunta 3. $g(x) = \mathrm{H}(x)\cos(x)$

$$(f * g)(x) = \mathrm{H}(x - 1)[x - 1 - \mathrm{sen}(x - 1)] + \mathrm{H}(x)\underbrace{\int_0^\infty xe^{-2x}\,\mathrm{sen}\,x\,dx}_{(2)}$$

Para (2):

$$\int_0^\infty xe^{-2x}\,\mathrm{sen}\,x\,dx = \int_{-\infty}^{+\infty} \mathrm{H}(x)\,xe^{-2x}\,\mathrm{sen}\,x\,dx$$

Recordemos que la Transf. de Laplace viene dada por:

$$\mathscr{L}(h(x))(z) = \int_{-\infty}^{+\infty} \mathrm{H}(x)\,xe^{-zx}\,dx$$

Así, es nuestro caso $z = 2$ y $h(x) = x\mathrm{H}(x)\,\mathrm{sen}(x)$, por lo tanto:

$$\int_0^\infty xe^{-2x}\operatorname{sen} x\, dx = \mathscr{L}\left(x\mathrm{H}(x)\operatorname{sen} x\right)(z=2) = -\frac{d}{dz}\,\left(\mathscr{L}\left(x\mathrm{H}(x)\operatorname{sen} x\right)\right)(z=2)$$

$-\dfrac{d}{dz}\,\left(\dfrac{1}{z^2+1}\right)\Bigg|_{z=2} = \dfrac{2z}{(1+z^2)^2}\Bigg|_{z=2} = \dfrac{4}{25}.$ Sustituyendo en la ecuación inicial obtenemos:

$$(f*g)(x) = \mathrm{H}(x-1)x - \mathrm{H}(x-1) - \mathrm{H}(x-1)\operatorname{sen}(x-1) + \frac{4}{25}\,\mathrm{H}(x)$$

$$(f*\mathrm{H}(x)\cos(x)) = \mathrm{H}(x-1)x - \mathrm{H}(x-1) - \mathrm{H}(x-1)\operatorname{sen}(x-1) + \frac{4}{25}\,\mathrm{H}(x)$$

Aplicando Transf. de Laplace a ambos lados de la ecuación y tomando $f(x) \xrightarrow{\mathscr{L}} \mathrm{F}(z)$, obtenemos:

$$\mathscr{L}\left(f(x)\right)(z)\mathscr{L}\left(\mathrm{H}(x)\cos x\right)(z) = \mathscr{L}\left(x\mathrm{H}(x-1)\right)(z) - \mathscr{L}\left(\mathrm{H}(x-1)\right)(z) +$$

$$-\mathscr{L}\left(\mathrm{H}(x-1)\operatorname{sen}(x-1)\right)(z) + \frac{4}{25}\mathscr{L}\left(\mathrm{H}(x)\right)(z)$$

$$\mathrm{F}(z)\left(\frac{z}{z^2+1}\right) = -\frac{d}{dz}\left(\frac{e^{-z}}{z}\right) - \frac{e^{-z}}{z} - \frac{e^{-z}}{z^2+1} + \frac{4}{25}\left(\frac{1}{z}\right)$$

$$\mathrm{F}(z)\left(\frac{z}{z^2+1}\right) = -\left(\frac{(-e^{-z})(z) - e^{-z}}{z^2}\right) - \frac{e^{-z}}{z} - \frac{e^{-z}}{z^2+1} + \frac{4}{25}\cdot\frac{1}{z}$$

Simplificando y despejando $\mathrm{F}(z)$:

$$\mathrm{F}(z) = \frac{e^{-z}}{z^3} + \frac{4}{25}\left(1 + \frac{1}{z^2}\right)$$

Tomando la Transf. inversa, $\mathrm{F}(z) \xrightarrow{\mathscr{L}^{-1}} f(x)$

$$\mathscr{L}^{-1}\left(\mathrm{F}(z)\right)(x) = \mathscr{L}^{-1}\left(\frac{e^{-z}}{z^3}\right)(x) + \frac{4}{25}\,\mathscr{L}^{-1}\left(1 + \frac{1}{2}\right)(x)$$

Así,

$$\boxed{f(x) = \frac{1}{2}\,\mathrm{H}(x-1)(x-1)^2 + \frac{4}{25}(\delta(x) + x\mathrm{H}(x))}$$